Journal of Approximation Theory **102**, 96–119 (2000)
Article ID jath.1999.3383, available online at http://www.idealibrary.com on IDEAL®

Orthogonal Polynomials on the Circumference and Arcs of the Circumference[1]

Franz Peherstorfer

Institut für Analysis und Numerik, Johannes Kepler Universität Linz, 4040 Linz Auhof, Austria
E-mail: franz.peherstorfer@jk.uni-linz.ac.at.

and

Robert Steinbauer[2]

Institut für Analysis und Numerik, Johannes Kepler Universität Linz, 4040 Linz Auhof, Austria
E-mail: robert.steinbauer@jk.uni-linz.ac.at

Communicated by Guillermo López Lagomasino

Received September 15, 1998; accepted in revised form March 25, 1999

In this paper we study measures and orthogonal polynomials with asymptotically periodic reflection coefficients. It's known that the support of the orthogonality measure of such polynomials consists of several arcs. We show how the measure of orthogonality can be approximated (resp. described) by the aid of the related orthonormal polynomials if the reflection coefficients are additionally of bounded variation (mod N). As an interesting byproduct we obtain that the orthogonality measure is (up to N points) absolutely continuous on the whole circumference, if the reflection coefficients $\{a_n\}$ are of bounded variation (mod N) and satisfy $\lim_{n\to\infty} a_n = 0$. Furthermore, it is demonstrated that the reflection coefficients remain asymptotically periodic if point measures are added on the support. Finally, we prove that under certain conditions on the arcs orthogonality measures which satisfy a generalized Szegő condition have asymptotically periodic reflection coefficients. © 2000 Academic Press

AMS 1991 subject classifications: 33C45, 42C05.

Key Words: orthogonal polynomials; unit circle; arcs; measures; asymptotically periodic reflection coefficients.

[1] This work was supported by the Austrian Fonds zur Förderung der wissenschaftlichen Forschung, project-number P12985-TEC, and by a MAX-KADE postdoctoral fellowship, selected by the Österreichischen Akademie der Wissenschaften.

[2] This work was started while the second author was visiting the Department of Mathematics at Ohio State University, U.S.A.

0021-9045/00 $35.00

1. INTRODUCTION AND NOTATION

Monic orthogonal polynomials $P_n(z, \sigma) = z^n + \ldots$, $n \in \mathbb{N}_0$, on the unit circle with respect to a nonnegative measure σ are defined by

$$\int_0^{2\pi} e^{-ij\varphi} P_n(e^{i\varphi}, \sigma)\, d\sigma(\varphi) = 0 \qquad \text{for} \quad j = 0, 1, \ldots, n-1 \tag{1.1}$$

and satisfy the recurrence relation

$$P_{n+1}(z, \sigma) = zP_n(z, \sigma) + a_n P_n^*(z, \sigma), \qquad n \in \mathbb{N}_0, \qquad P_0(z, \sigma) = 1, \tag{1.2}$$

where the parameters $a_n := a_n(\sigma) := P_{n+1}(0, \sigma)$ are called reflection coefficients and satisfy $|a_n| < 1$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here, $P_n^*(z, \sigma) := z^n \overline{P_n(1/\bar{z}, \sigma)}$ is the reversed polynomial.

Throughout this paper we will always assume that σ is normalized by $\sigma([0, 2\pi]) = 2\pi$ and that its support contains infinitely many points.

It will turn out that it is often more natural to deal with the orthonormal polynomials

$$\Phi_n(z, \sigma) := \frac{P_n(z, \sigma)}{\sqrt{d_n}} = \frac{z^n}{\sqrt{d_n}} + \ldots, \qquad n \in \mathbb{N}_0, \tag{1.3}$$

where $d_n := \prod_{j=0}^{n-1} (1 - |a_j|^2)$, which satisfy

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{i\varphi}, \sigma)\, \overline{\Phi_m(e^{i\varphi}, \sigma)}\, d\sigma(\varphi) = \delta_{nm}.$$

In this paper we study orthogonality measures and orthogonal polynomials, whose reflection coefficients are asymptotically periodic and of bounded variation (modulo$N$), where $N \in \mathbb{N}$ is fixed. This means: there exist values $a_0^0, \ldots, a_{N-1}^0$, $|a_j^0| < 1$, such that

$$\lim_{\nu \to \infty} a_{\nu N + j} = a_j^0 \qquad \text{for} \quad j = 0, 1, \ldots, N-1, \tag{1.4}$$

and

$$\sum_{n=0}^{\infty} |a_n - a_{n+N}| < \infty. \tag{1.5}$$

In [12, 14] the authors have investigated the asymptotic behaviour of the "asymptotically periodic" orthogonal polynomials $\Phi_n(z, \sigma)$ for $n \to \infty$ outside the support of the measure of orthogonality and in [15] also on the support. The reader should also compare the recent papers by

Bello/López [2] and Barrios/López [3], where ratio asymptotics of the orthogonal polynomials are given. In this contribution we deal with the question how the orthogonality measure respectively its absolutely continuous part f can be described with the aid of the related orthogonal polynomials. As an interesting byproduct we obtain that the orthogonality measure is (up to N points) absolutely continuous on the circumference, if the reflection coefficients $\{a_n\}$ are of bounded variation (mod N) and satisfy $\lim_{n\to\infty} a_n=0$. Furthermore, it is shown that, under certain conditions, adding point measures on the support does not disturb the asymptotic behaviour of the reflection coefficients. Finally, heavily based on the results of Widom [18], we prove that polynomials orthogonal with respect to weight functions which satisfy a generalized Szegő condition have asymptotically periodic reflection coefficients, if there exists a so-called T-polynomial on the arcs. But first of all, in Section 2, we will give some basic properties of the "periodic" orthogonal polynomials $P_n(z, \sigma_0)$ generated by the periodic sequence of reflection coefficients $\{a_n^0\}$, $a_{n+N}^0=a_n^0$, which will be essential in the proofs of our results. In Section 4 we will give the proofs.

2. THE KNOWN PERIODIC CASE

In this section we collect properties of orthogonal polynomials, which are generated by the sequence $\{a_n^0\}$ of periodic reflection coefficients,

$$a_n^0=a_{n+N}^0, \qquad n\in\mathbb{N}_0, \qquad N\in\mathbb{N} \text{ fixed},$$

and denoted by $P_n(z, \sigma_0)$, i.e., σ_0 is the corresponding orthogonality measure. Such polynomials resp. measures have mainly been studied by Geronimus [4, 5, 6] and in the last years also by the authors [11, 12] and we will mainly refer to the latter paper.

It is known that the support of σ_0 consists of l, $l\leqslant N$, disjoint subintervals of $[0, 2\pi]$ and at most of a finite number of points outside the intervals. Let us denote these intervals by

$$E_l := \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}], \tag{2.1}$$

where the φ_k's, $k=1, ..., 2l$, are pairwise distinct. For the corresponding arcs on the unit circle we write

$$\Gamma_{E_l} := \{e^{i\varphi} : \varphi\in E_l\}.$$

The set E_l and the measure σ_0 can completely be described by the orthogonal polynomials themselves in the following way: Let $\{\Omega_n(z, \sigma_0)\}$

be the monic polynomials of the second kind corresponding to $P_n(z, \sigma_0)$, which are recursively given by

$$\Omega_{n+1}(z, \sigma_0) := z\Omega_n(z, \sigma_0) - a_n^0 \Omega_n^*(z, \sigma_0), \qquad \Omega_0(, \sigma_0) := 1.$$

Note the opposite sign in front of a_n^0.

Next we define the value

$$L := 2\left(\prod_{j=0}^{N-1} (1 - |a_j^0|^2)\right)^{1/2} = 2\sqrt{d_N^0} \tag{2.2}$$

and the monic polynomials

$$\begin{aligned} \mathcal{T}(z) &:= \tfrac{1}{2}(P_N(z, \sigma_0) + \Omega_N(z, \sigma_0) + P_N^*(z, \sigma_0) + \Omega_N^*(z, \sigma_0)) = z^N + \cdots \\ \mathfrak{R}(z) &:= R(z)\, \mathcal{U}^2(z) := \mathcal{T}^2(z) - L^2 z^N = z^{2N} + \cdots . \end{aligned} \tag{2.3}$$

Then it can be shown that $\mathfrak{R}$ has all its zeros on $|z| = 1$. The selfreversed polynomial R is of degree $2l$ and vanishes exactly at the boundary points $e^{i\varphi_j}$, $j = 1, ..., 2l$, of the arcs. Moreover, there are exactly $N - l$ ($N = l$ is possible) double roots in $\{e^{i\varphi}: \varphi \in \operatorname{int} E_l\}$. Thus, in (2.3) the polynomial $\mathcal{U}$ is selfreversed, of degree $N - l$, and vanishes exactly at the double zeros of $\mathfrak{R}$. Now, the set E_l can be expressed with the aid of the polynomials R and $\mathcal{T}$, respectively, by

$$E_l = \{\varphi \in [0, 2\pi] : e^{-il\varphi} R(e^{i\varphi}) \leqslant 0\} = \{\varphi \in [0, 2\pi] : |\mathcal{T}(e^{i\varphi})| \leqslant L\}. \tag{2.4}$$

Note that R is a selfreversed polynomial; thus $e^{-il\varphi} R(e^{i\varphi})$ is a real trigonometric polynomial.

Let us also point out that it can be shown with the help of (2.3) that

$$\left| \frac{\mathcal{T}(z) + \sqrt{R(z)}\, \mathcal{U}(z)}{L} \right| = 1 \qquad \text{for all} \quad z \in \Gamma_{E_l}, \tag{2.5}$$

whereas

$$\left| \frac{\mathcal{T}(z) + \sqrt{R(z)}\, \mathcal{U}(z)}{L} \right| > 1 \qquad \text{on } \mathbb{C} \setminus \Gamma_{E_l};$$

compare also [12, Lemma 3.1].

The absolutely continuous part f_0 of σ_0 is given explicitly in terms of the corresponding orthogonal polynomials by

$$f_0(\varphi)=\begin{cases}\left|\dfrac{\sqrt{R(e^{i\varphi})}}{V(e^{i\varphi})\,A(e^{i\varphi})}\right|, & \varphi\in E_l\\ 0, & \varphi\notin E_l,\end{cases} \tag{2.5}$$

where

$$\mathfrak{A}(z):=V(z)\,A(z):=\frac{P_N^*(z,\sigma_0)-P_N(z,\sigma_0)}{\mathscr{U}(z)}\in\mathbb{P}_l. \tag{2.6}$$

Here, the selfreversed polynomial V contains exactly those zeros of $\mathfrak{A}$, which lie on the set $\{e^{i\varphi_j}: j=1, ..., 2l\}$. All the zeros of the polynomial A are outside Γ_{E_l}.

The singular part of σ_0 consists of at most a finite number of mass points and as far as they appear they are located outside E_l at (some of the) zeros of A; to be more precise, at points ξ where $A(e^{i\xi})=0$.

In order to state our results on the "asymptotically periodic" measure σ, it will be useful to introduce also the following notation: Since $\mathscr{T}$, $\mathscr{U}$, and R are selfreversed polynomials the settings

$$\begin{aligned}\tau(\varphi)&:=e^{-i(N/2)\varphi}\,\mathscr{T}(e^{i\varphi})\\ u(\varphi)&:=e^{-i((N-l)/2)\varphi}\,\mathscr{U}(e^{i\varphi})\\ \mathscr{R}(\varphi)&:=e^{-il\varphi}\,R(e^{i\varphi})\end{aligned} \tag{2.7}$$

$\varphi\in[0,2\pi]$, give real trigonometric polynomials. Further, let

$$r(\varphi):=\begin{cases}ie^{-i(l/2)\varphi}\sqrt{R(e^{i\varphi})}=(-1)^{j+1}\sqrt{|\mathscr{R}(\varphi)|}, & \text{for}\quad \varphi\in[\varphi_{2j-1},\varphi_{2j}]\\ e^{-i(l/2)\varphi}\sqrt{R(e^{i\varphi})}=(-1)^{j}\sqrt{|\mathscr{R}(\varphi)|}, & \text{for}\quad \varphi\in[\varphi_{2j},\varphi_{2j+1}],\end{cases} \tag{2.8}$$

with $\varphi_0:=0$ and $\varphi_{2l+1}:=2\pi$, be a real continuous square-root function, which changes sign from the interval $[\varphi_{2j-1},\varphi_{2j}]$ to the interval $[\varphi_{2j+1},\varphi_{2j+2}]$.

With this notation the function f_0 from (2.5) can also be written as

$$f_0(\varphi)=\begin{cases}\dfrac{r(\varphi)}{\mathscr{V}(\varphi)\,\mathscr{A}(\varphi)}\geqslant 0, & \varphi\in E_l\\ 0, & \varphi\notin E_l,\end{cases} \tag{2.9}$$

where $(\mathcal{V}\mathcal{A})(\varphi) := ie^{-i(l/2)\varphi}(VA)(e^{i\varphi})$ is again a real trigonometric polynomial.

3. APPROXIMATING THE MEASURE σ

In (2.5) we have seen, how the absolutely continuous part f_0 of the "periodic" measure σ_0 can be represented with the help of the corresponding orthonormal polynomials $\Phi_n(z, \sigma_0)$. Naturally the question arises, if there holds a similar representation for "asymptotically periodic" measures σ, or in other words, is it possible to describe the orthogonality measure respectively its absolutely continuous part f with the aid of the related orthogonal polynomials?

If condition (1.4) is satisfied then the accumulation points of $\text{supp}(\sigma)$ and $\text{supp}(\sigma_0)$ coincide, i.e.,

$$(\text{supp}(\sigma))' = (\text{supp}(\sigma_0))'. \tag{3.1}$$

For $N=1$ this fact has been proved in [9, Theorem 3]. The proof given in [9] can easily be extended to the general case $N \in \mathbb{N}$. Hence, the support $\text{supp}(\sigma)$, where σ denotes the perturbed measure in the sense of (1.4), also consists of the l intervals E_l and at most a denumerable number of points in $[0, 2\pi)$ outside the intervals. Moreover, the end-points of E_l, i.e., $\varphi_1, ..., \varphi_{2l}$, are the only possible accumulation points of the mass points, which all lie outside of E_l.

We begin with the following definition:

$$\begin{aligned}\Theta_n(z) &:= \frac{iL}{2z^{n+l/2}\mathcal{U}(z)} \begin{vmatrix} \Phi_n(z,\sigma) & \Phi_{n+N}(z,\sigma) \\ \Phi_n^*(z,\sigma) & \Phi_{n+N}^*(z,\sigma) \end{vmatrix} \\ &= \frac{iL(\Phi_n(z,\sigma)\,\Phi_{n+N}^*(z,\sigma) - \Phi_n^*(z,\sigma)\,\Phi_{n+N}(z,\sigma))}{2z^{n+l/2}\,\mathcal{U}(z)}, \qquad n \in \mathbb{N}_0.\end{aligned} \tag{3.2}$$

Let us point out that for $\sigma = \sigma_0$

$$\Theta_n(z) = iz^{-l/2}V(z)\,A(z) \qquad \text{and} \qquad \Theta_n(e^{i\varphi}) = \mathcal{V}(\varphi)\,\mathcal{A}(\varphi);$$

recall (2.5) and (2.9). Hence, we expect that for σ "close" to σ_0 and for sufficiently large n the function $\Theta_n(e^{i\varphi})$ will describe the weight function on the support.

Applying the recurrence relation (1.2) several times, the function Θ_n can be expanded in the series (compare [12, formula (4.10)])

$$\Theta_n(z)=\frac{iL\lambda_n}{2z^{-(N-1)/2}\mathcal{U}(z)}\left(\frac{\Phi_N^*-\Phi_N}{z^{N/2}}+\sum_{\nu=0}^{n-1}\frac{\beta_\nu}{z^{\nu+1+N/2}}\left\{(a_\nu-a_{\nu+N})\,\Phi_\nu^*\Phi_{\nu+N}^*\right.\right.$$
$$\left.\left.+z(a_\nu\bar{a}_{\nu+N}-\bar{a}_\nu a_{\nu+N})\,\Phi_\nu^*\Phi_{\nu+N}-z^2(\bar{a}_\nu-\bar{a}_{\nu+N})\,\Phi_\nu\Phi_{\nu+N}]\right\}\right), \tag{3.3}$$

where Φ_ν stands for $\Phi_\nu(z,\sigma)$ and where

$$\lambda_n:=\prod_{j=0}^{n-1}\left(\frac{1-\bar{a}_j a_{j+N}}{1-|a_j|^2}\right),\qquad \beta_\nu:=\frac{1}{1-|a_\nu|^2}\prod_{j=0}^{\nu}\left(\frac{1-|a_j|^2}{1-\bar{a}_j a_{j+N}}\right).$$

By its definition,

$$\vartheta_n(\varphi):=\Theta_n(e^{i\varphi})$$

is a real trigonometric polynomial, which coincides with $\mathcal{V}(\varphi)\,\mathcal{A}(\varphi)$ if $\sigma=\sigma_0$. As the following theorem shows ϑ_n approximates the absolutely continuous part of σ.

THEOREM 1. *Let the assumptions* (1.4) *and* (1.5) *be satisfied and let us denote the absolutely continuous part of σ by f. The function r is given as in* (2.8). *Then f vanishes outside E_l and there holds*

$$\lim_{n\to\infty}\vartheta_n(\varphi)=:\vartheta(\varphi)=\frac{r(\varphi)}{f(\varphi)} \tag{3.4}$$

uniformly on compact subsets of int $E_l\backslash\{\psi_1,...,\psi_{N-l}\}$, *where the ψ_j's are the zeros of $u(\varphi)$; compare* (2.7) *and the definition of $\mathcal{U}$ in* (2.3). *Furthermore, σ is absolutely continuous on* int $E_l\backslash\{\psi_1,...,\psi_{N-l}\}$ *and f is positive and continuous there.* ▮

Remark. Under the stronger assumption (3.7), see below, the limit relation (3.4) follows immediately from [12, Theorem 4.1][3], even uniformly compact on int E_l. It remains to be shown that the result also holds true under the weaker condition (1.5).

From the above theorem we can also derive a result for the unit circle.

[3] Correction to Theorem 4.1 in [12]. In the Theorem the phrase "where $\alpha(\varphi)$ is a differentiable function on int E_l" has to be replaced by "where $\alpha(\varphi)$ is a continuous function on int E_l".

COROLLARY 1. *Let* $\{a_n = P_{n+1}(0, \sigma)\}_{n \in \mathbb{N}_0}$ *be a sequence of reflection coefficients from Nevai's class, i.e.,* $\lim_{n \to \infty} a_n = 0$. *Further suppose that there exists a positive integer N such that condition* (1.5) *holds. Then the corresponding measure* σ *is absolutely continuous on* $[0, 2\pi] \backslash \{2\pi k/N: k = 0, ..., N-1\}$, $f(\varphi) = \sigma'(\varphi)$ *is positive and continuous there and can be written as*

$$f(\varphi) = \frac{2 \sin(N/2)\, \varphi}{\vartheta(\varphi)}, \qquad \varphi \in [0, 2\pi] \backslash \left\{\frac{2\pi k}{N} : k = 0, ..., N-1\right\},$$

where the real function ϑ *is given as in Theorem* 1. ∎

Remark. In particular, from Corollary 1 we see that there are no mass points in $[0, 2\pi] \backslash \{2\pi k/N: k = 0, ..., N-1\}$ if $a_n \to 0$ and if (1.5) is satisfied. For instance, (1.5) is a consequence of the Geronimus' condition $\sum_{n=0}^{\infty} |a_n| < \infty$ and this latter condition implies that σ is absolutely continuous on $[0, 2\pi]$. But obviously (1.5) is much weaker than Geronimus' condition, as the examples $\{a_n = 1/n\}$ or $\{a_n = 1/\sqrt{n}\}$ show (the last sequence is even outside the Szegő class).

Let us also give another method of representing the absolutely continuous part of σ with the aid of the orthonormal polynomials. Therefore, we define the following functions (for the motivation of these definitions compare the proof of Theorem 2 below):

$$\begin{aligned}\mathcal{S}_{1,n}(z) = \frac{\mu_n}{2i} \sum_{\nu=0}^{n-1} \frac{\kappa_\nu}{z^{\nu+1}} \Big\{ & z(a_\nu^0 \bar{a}_\nu - \bar{a}_\nu^0 a_\nu)\, \Phi_\nu^*(z, \sigma_0)\, \Phi_\nu(z, \sigma) \\ & = -z^2(\bar{a}_\nu^0 - \bar{a}_\nu)\, \Phi_\nu(z, \sigma_0)\, \Phi_\nu(z, \sigma) + (a_\nu^0 - a_\nu)\, \Phi_\nu^*(z, \sigma_0)\, \Phi_\nu^*(z, \sigma) \Big\}\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{S}_{2,n}(z) = \frac{\bar{\mu}_n}{2} \sum_{\nu=0}^{n-1} \frac{\bar{\kappa}_\nu}{z^{\nu+1}} \Big\{ & z(a_\nu^0 \bar{a}_\nu - \bar{a}_\nu^0 a_\nu)\, \Psi_\nu(z, \sigma_0)\, \Phi_\nu^*(z, \sigma) \\ & = -z^2(\bar{a}_\nu^0 - \bar{a}_\nu)\, \Psi_\nu(z, \sigma_0)\, \Phi_\nu(z, \sigma) - (a_\nu^0 - a_\nu)\, \Psi_\nu^*(z, \sigma_0)\, \Phi_\nu^*(z, \sigma) \Big\}\end{aligned} \tag{3.6}$$

where

$$\mu_n = \prod_{j=0}^{n-1} \frac{1 - \bar{a}_j^0 a_j}{\sqrt{(1 - |a_j^0|^2)(1 - |a_j|^2)}}$$

$$\kappa_\nu = \frac{1}{\sqrt{(1 - |a_\nu^0|^2)(1 - |a_\nu|^2)}} \prod_{j=0}^{\nu} \frac{\sqrt{(1 - |a_j^0|^2)(1 - |a_j|^2)}}{1 - \bar{a}_j^0 a_j}.$$

The following theorem gives an alternative representation of the absolutely continuous part of σ, where instead of condition (1.5) the stronger condition (3.7) is needed.

THEOREM 2. *Suppose that the reflection coefficients* $\{a_n = P_{n+1}(0, \sigma)\}$ *from* (1.4) *converge sufficiently fast such that the condition*

$$\sum_{n=0}^{\infty} |a_n - a_n^0| < \infty \tag{3.7}$$

is satisfied. Then the limits

$$\mathcal{S}_1(z) := \lim_{n \to \infty} \mathcal{S}_{1,n}(z) \qquad \text{and} \qquad \mathcal{S}_2(z) := \lim_{n \to \infty} \mathcal{S}_{2,n}(z) \tag{3.8}$$

exist uniformly on compact subsets of $\{e^{i\varphi} : \varphi \in \operatorname{int} E_l\}$. *Let* f *denote the absolutely continuous part of the measure* σ. *Then* σ *is absolutely continuous on* $\operatorname{int} E_l$, f *is positive and continuous there and for all* $\varphi \in \operatorname{int} E_l$ *there holds*

$$iz^{-l/2} V(z)[C(z)\, \mathcal{S}_1^2(z) + 2iB(z)\, \mathcal{S}_1(z)\, \mathcal{S}_2(z) + A(z)\, \mathcal{S}_2^2(z)]$$
$$= \frac{r(\varphi)}{f(\varphi)}, \qquad z = e^{i\varphi}, \tag{3.9}$$

where the function r *is given as in* (2.8), *the polynomial* A *as in* (2.6), *and the polynomials* B *and* C *by*

$$B(z) = \frac{\Omega_N(z, \sigma_0) + \Omega_N^*(z, \sigma_0) - P_N(z, \sigma_0) - P_N^*(z, \sigma_0)}{2V(z)\, \mathcal{U}(z)}$$

$$C(z) = \frac{\Omega_N^*(z, \sigma_0) - \Omega_N(z, \sigma_0)}{V(z)\, \mathcal{U}(z)}. \quad \blacksquare$$

Suppose that the reflection coefficients $\{a_n\}$ associated with the orthogonality measure σ satisfy Szegő's condition $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. It is well known (see e.g. [8, 16]) that this is equivalent to Szegő's condition $\int_0^{2\pi} \log f(\varphi)\, d\varphi > -\infty$ on the orthogonality measure σ, where f denotes the absolutely continuous part of σ. Hence, if we add point measures to such a measure σ the new recurrence coefficients, denoted by $(\tilde{a}_n)$, will also satisfy the Szegő condition and therefore the limit relations $\lim_{n \to \infty} \tilde{a}_n = \lim_{n \to \infty} a_n = 0$. Let us present a similar result for several arcs.

THEOREM 3. *Let* σ *be a measure whose reflection coefficients* $\{a_n(\sigma)\}$ *are asymptotically periodic and satisfy* (1.5). *Furthermore, let* $\{\alpha_j\}_{j=1}^{M}$ *be* *given points from* int $E_l \backslash \{\psi_1, ..., \psi_{N-l}\}$ *(recall the definition in Theorem* 1) *and define the new measure by*

$$\mu(\varphi) := c\left(\sigma(\varphi) + \sum_{j=1}^{M} \lambda_j \delta_{\alpha_j}(\varphi)\right), \qquad \lambda_j \geqslant 0, \qquad c > 0.$$

Here, δ_{α_j} *denotes the Dirac-measure with mass at* α_j *and* c *is the normalization factor such that* $\mu([0, 2\pi]) = 2\pi$. *Then the reflection coefficients* $\{a_n(\mu)\}$ *associated with the measure* μ *are again asymptotically periodic and there holds*

$$\lim_{n\to\infty} (a_n(\mu) - a_n(\sigma)) = 0. \quad \blacksquare \tag{3.10}$$

Remark. Note that the bounded variation condition (1.5) is not preserved in general for the modified measure μ.

Remark. (a) Under the stronger assumption (3.7) the statement of Theorem 3 holds true for all added mass points from int E_l, i.e., there are no forbidden points in E_l. This follows from the uniform boundedness of the orthonormal polynomials on compact subsets of int E_l (see [12, Cor. 2.2]).

(b) Let us point out that a limit relation such as (3.10) does not hold true in general, if the mass points α_j are chosen outside the set E_l as the following simple example shows: Let

$$a_n(\sigma) := \frac{1}{\sqrt{2}} \qquad \text{and} \qquad a_n(\mu) := -\frac{1}{\sqrt{2}}$$

for all $n \in \mathbb{N}_0$. Then, obviously, $a_n(\sigma) - a_n(\mu) \nrightarrow 0$ as $n \to 0$. But the measures σ and μ only differ by a mass point at $\varphi = 0$: From [6] or from [11] one can show that σ is absolutely continuous with

$$d\sigma(\varphi) = \sigma'(\varphi)\, d\varphi = \begin{cases} \dfrac{1}{\sqrt{2}-1} \dfrac{\sqrt{-\cos\varphi}}{\sin\varphi/2}\, d\varphi, & \varphi \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{0}\right] \\ 0, & \text{else} \end{cases}$$

and

$$\mu(\varphi) = \frac{\sqrt{2}-1}{\sqrt{2}+1}\left(\sigma(\varphi) + \frac{4\pi}{\sqrt{2}-1}\,\delta_0(\varphi)\right).$$

Recall that $\varphi = 0 \notin E_1 = [\pi/2, 3\pi/2]$. Concerning point measures see also [13].

Finally, we also would like to state the following theorem which gives a Szegő-type result for arcs of the unit circle and which has a reverse-character in that sense that it starts from properties of the orthogonality measure and gives information about the corresponding reflection coefficients. In [11] we have shown that the existence of a so-called T-polynomial $\mathcal{T}$ (compare the definition (3.12) below) on the arcs

$$\Gamma := \bigcup_{j=1}^{l} \Gamma_j \qquad \text{with} \quad \Gamma_j := [e^{i\varphi_{2j-1}}, e^{i\varphi_{2j}}],$$

$\varphi_1 < \varphi_2 < \cdots < \varphi_{2l} < \varphi_1 + 2\pi$, implies that weight functions of the form (recall (2.5))

$$\left| \frac{\sqrt{R(e^{i\varphi})}}{V(e^{i\varphi})\, A(e^{i\varphi})} \right| \quad \text{on the arcs} \quad \text{and zero elsewhere} \tag{3.11}$$

have periodic reflection coefficients. A selfreversed polynomial $\mathcal{T}$ of degree N, $N \geqslant l$, is called a *T-polynomial* on Γ, if it satisfies the condition (compare also the second line in (2.3))

$$\mathcal{T}^2(z) - R(z)\, \mathcal{U}^2(z) = L^2 z^N, \tag{3.12}$$

where $\mathcal{U}$ is a selfreversed polynomial of degree $N - l$ and where L is a positive constant. Therefore, we expect that suitable "perturbations" of weight functions of the form (3.11) will lead to asymptotically periodic reflection coefficients.

Notation. We say that $\Gamma = \bigcup_{j=1}^{l} \Gamma_j$ belongs to the class $\mathcal{P}(N)$ if there exists a T-polynomial $\mathcal{T}$ of degree N, $N \geqslant l$, which satisfies (3.12).

Let us point out that we have proved in [14] that condition (3.12), i.e., $\Gamma \in \mathcal{P}(N)$, is equivalent to the fact that the harmonic measure $\omega(\Gamma_j, \infty)$ of every arc Γ_j gives a rational number of the form k_j/N. Recall the definition of the harmonic measure of Γ_j at ∞:

$$\omega(\Gamma_j, \infty) = \frac{1}{2\pi} \oint_{\Gamma_j} \frac{\partial}{\partial n_\xi}\, g(\xi)\, |d\xi|,$$

where $(\partial/\partial n_\xi)$ is the normal derivative at ξ and where $g(\xi) := g(\xi, \infty)$ denotes the (real) Green's function for the set $\overline{\mathbb{C}} \backslash \Gamma$ with pole at ∞.

Based on results of Widom [18] we are now able to show

THEOREM 4. *Assume that the union of the l disjoint arcs $\Gamma = \bigcup_{j=1}^{l} [e^{i\varphi_{2j-1}}, e^{i\varphi_{2j}}]$, $\varphi_1 < \varphi_2 < \cdots < \varphi_{2l} < \varphi_1 + 2\pi$, belongs to $\mathcal{P}(N)$, $N \in \mathbb{N} \backslash \{1, ..., l-1\}$, and put $E_l := \{\varphi : e^{i\varphi} \in \Gamma\}$. Further suppose that*

$d\sigma(\varphi) = f(\varphi)\,d\varphi$ is a positive and absolutely continuous measure on E_l and that $f(\varphi)$ satisfies the generalized Szegő condition $\int_{\varphi_{2j-1}}^{\varphi_{2j}} \log f(\varphi)/\sqrt{\sin((\varphi-\varphi_{2j-1})/2)\sin((\varphi_{2j}-\varphi)/2)}\,d\varphi > -\infty$ for $j=1, ..., l$. Then the reflection coefficients $\{a_n(\sigma)\}$ are asymptotically periodic, i.e., there exist values $a_0^0, ..., a_{N-1}^0 \in \mathbb{C}$, $|a_k^0| < 1$ for $k=0, ..., N-1$, such that

$$\lim_{m\to\infty} a_{k+mN}(\sigma) = a_k^0 \qquad \text{for} \quad k=0, ..., N-1. \tag{3.13}$$

Let us mention that the polynomial $P_N(z, \sigma_0)$ generated by the reflection coefficients $a_0^0, ..., a_{N-1}^0$ from (3.13) and its polynomial of the second kind $\Omega_N(z, \sigma_0)$ give the T-polynomial $\mathcal{T}$ on Γ by the relation

$$\mathcal{T}(z) = \tfrac{1}{2}(P_N(z, \sigma_0) + \Omega_N(z, \sigma_0) + P_N^*(z, \sigma_0) + \Omega_N^*(z, \sigma_0)).$$

This follows from (3.1) and [11, Theorem 4.3] (compare also (2.3)).

4. PROOFS

Let us begin with the following

DEFINITION. The monic k-associated polynomials, resp. the monic associated polynomials of the second kind, $k \in \mathbb{N}_0$, are given by the shifted recurrence formula

$$P_{n+1}^{(k)}(z, \sigma) := zP_n^{(k)}(z, \sigma) + a_{n+k}P_n^{(k)*}(z, \sigma), \qquad P_0^{(k)}(z, \sigma) := 1$$

$$\Omega_{n+1}^{(k)}(z, \sigma) := z\Omega_n^{(k)}(z, \sigma) + a_{n+k}\Omega_n^{(k)*}(z, \sigma), \qquad \Omega_0^{(k)}(z, \sigma) := 1.$$

For $k=0$ we simply write again $P_n^{(0)} = P_n$ and $\Omega_n^{(0)} = \Omega_n$, respectively.

Proof of Theorem 1. If we can show that the polynomials $\Phi_n(e^{i\varphi}, \sigma)$ are uniformly bounded on compact subsets of int $E_l \backslash \{\psi_1, ..., \psi_{N-l}\}$ then by (1.5) and (3.3) the limit

$$\vartheta(\varphi) := \lim_{n\to\infty} \vartheta_n(\varphi) \tag{4.1}$$

exists uniformly compact on int $E_l \backslash \{\psi_1, ..., \psi_{N-l}\}$. For the proof of the boundedness of the orthonormal polynomials $\Phi_n(z, \sigma)$ we follow some ideas given in [12, Lemma 3.2 and Theorem 3.4]: For the rest of the proof we will write $P_n(z)$, $\Omega_n(z)$, etc. instead of $P_n(z, \sigma)$, $\Omega_n(z, \sigma)$, etc.. Motivated by (2.2) and (2.3), respectively, let us define the polynomials, $n \in \mathbb{N}_0$,

$$\mathscr{T}^{[n]}(z) := \tfrac{1}{2}(P_N^{(n)}(z) + \Omega_N^{(n)}(z) + P_N^{(n)*}(z) + \Omega_N^{(n)*}(z))$$

$$R^{[n]}(z)\, \mathscr{U}^{[n]\,2}(z) := \mathscr{T}^{[n]\,2}(z) - L^{[n]\,2} z^N,$$

$$L^{[n]} := 2\left(\prod_{j=0}^{N-1} (1 - |a_{n+j}|^2)\right)^{1/2},$$

where the $P_N^{(n)}$'s ($\Omega_N^{(n)}$'s) denotes the nth monic associated polynomials (of the second kind) and where $R^{[n]}$ has only simple zeros. Further, let the functions $y_\pm^{[n]}$ be given by

$$y_\pm^{[n]}(z) := \frac{\mathscr{T}^{[n]}(z) \pm \sqrt{R^{[n]}(z)}\, \mathscr{U}^{[n]}(z)}{L^{[n]}}, \qquad n \in \mathbb{N}_0,$$

where $\sqrt{R^{[n]}(e^{i\varphi})} := \lim_{s \to 1^-} \sqrt{R^{[n]}(se^{i\varphi})}$. Then, in a similar way as in the proof of [12, Lemma 3.2], one can derive the following relations

$$\begin{aligned}\Phi_{m+(v+2)N} &- y_\pm^{[m+(v+1)N]} \Phi_{m+(v+1)N} \\ &= \frac{L^{[m+vN]}}{L^{[m+(v+1)N]}}\, y_\mp^{[m+vN]}\, (\Phi_{m+(v+1)N} - y_\pm^{[m+vN]} \Phi_{m+vN}) \\ &\quad + \frac{2\delta^\pm_{m+vN}}{L^{[m+(v+1)N]}},\end{aligned}$$

where

$$\begin{aligned}\delta_n^\pm &= \{e_n + \tfrac{1}{2}(L^{[n]} y_\pm^{[n]} - L^{[n+N]} y_\pm^{[n+N]})\}\, \Phi_{n+N} + f_n \Phi^*_{n+N} \\ e_n &= \tfrac{1}{2}(P_N^{(n+N)} + \Omega_N^{(n+N)} - P_N^{(n)} - \Omega_N^{(n)}) \\ f_n &= \tfrac{1}{2}(P_N^{(n+N)} - \Omega_N^{(n+N)} - P_N^{(n)} + \Omega_N^{(n)}),\end{aligned}$$

and by iterating the above identity

$$\begin{aligned}\Phi_{m+(v+2)N} &- y_\pm^{[m+(v+1)N]} \Phi_{m+(v+1)N} \\ &= \frac{L^{[m]}}{L^{[m+(v+1)N]}} \left(\prod_{j=0}^{v} y_\mp^{[m+jN]}\right) [\Phi_{m+N} - y_\pm^{[m]} \Phi_m] \\ &\quad + \frac{2}{L^{[m+(v+1)N]}} \sum_{j=0}^{v} \left(\prod_{k+j+1}^{v} y_\mp^{[m+kN]}\right) \delta^\pm_{m+jN}.\end{aligned}$$

Hence,

$$\frac{2\sqrt{R^{[m+(v+1)N]}}\,\mathcal{U}^{[m+(v+1)N]}}{L^{[m+(v+1)N]}}\,\Phi_{m+(v+1)N}$$
$$=(\Phi_{m+(v+2)N}-y_-^{[m+(v+1)N]}\Phi_{m+(v+1)N})$$
$$-(\Phi_{m+(v+2)N}-y_+^{[m+(v+1)N]}\,\Phi_{m+(v+1)N})$$
$$=\frac{L^{[m]}}{L^{[m+(v+1)N]}}\left\{\left(y_+^{[m]}\prod_{j=0}^{v}y_-^{[m+jN]}-y_-^{[m]}\prod_{j=0}^{v}y_+^{[m+jN]}\right)\Phi_m\right.$$
$$\left.+\left(\prod_{j=0}^{v}y_+^{[m+jN]}-\prod_{j=0}^{v}y_-^{[m+jN]}\right)\Phi_{m+N}\right\}$$
$$+\frac{2}{L^{[m+(v+1)N]}}\sum_{j=0}^{v}\left(\delta_{m+jN}^-\prod_{k=j+1}^{v}y_+^{[m+kN]}\right.$$
$$\left.-\delta_{m+jN}^+\prod_{k=j+1}^{v}y_-^{[m+kN]}\right)$$

or equivalently

$$2\sqrt{R^{[m+vN]}}\,\mathcal{U}^{[m+vN]}\Phi_{m+(v+1)N}$$
$$=L^{[m]}\left\{\left(y_+^{[m]}\prod_{j=0}^{v}y_-^{[m+jN]}-y_-^{[m]}\prod_{j=0}^{v}y_+^{[m+jN]}\right)\Phi_m\right.$$
$$\left.+\left(\prod_{j=0}^{v}y_+^{[m+jN]}-\prod_{j=0}^{v}y_-^{[m+jN]}\right)\Phi_{m+N}\right\}$$
$$+2\sum_{j=0}^{v-1}\left(\delta_{m+jN}^-\prod_{k=j+1}^{v}y_+^{[m+kN]}-\delta_{m+jN}^+\prod_{k=j+1}^{v}y_-^{[m+kN]}\right). \qquad (4.2)$$

Let us now consider an arbitrary (but fixed) compact subset $\mathcal{E}$ of int $E_l \setminus \{\psi_1, ..., \psi_{N-l}\}$. Since $R^{[n]} \to R$, $\mathcal{U}^{[n]} \to \mathcal{U}$, and $\mathcal{T}^{[n]} \to \mathcal{T}$ uniformly compact on $\mathbb{C}$ as $n \to \infty$ we can choose an index $m = m(\mathcal{E})$ as large such that for all $n \geqslant m$

$$|R^{[n]}(e^{i\varphi})\,\mathcal{U}^{[n]2}(e^{i\varphi})| \geqslant \eta(\mathcal{E}) > 0 \qquad \text{and} \qquad |y_\pm^{[n]}(e^{i\varphi})| = 1$$

on $\{e^{i\varphi}: \varphi \in \mathcal{E}\}$; recall (2.4) and (2.5). Now, applying triangle-inequality to (4.2) gives

$$|\Phi_{m+(v+1)N}(z,\sigma)| \leqslant c_1(\mathcal{E}) + c_2(\mathcal{E}) \sum_{j=0}^{v-1}\left(\sum_{k=m+jN}^{m+(j+1)N-1}|a_{k+N}-a_k|\right)$$
$$\times|\Phi_{m+(j+1)N}(z)|$$

for all $z \in \{e^{i\varphi} : \varphi \in \mathscr{E}\}$, where $c_1(\mathscr{E})$ and $c_2(\mathscr{E})$ are positive constants only depending on the set $\mathscr{E}$ and where we used the well known identity $|\Phi_n(e^{i\varphi})| = |\Phi_n^*(e^{i\varphi})|$ for all $n \in \mathbb{N}$. Then from the discrete version of Gronwall's inequality (see e.g. [17, (2.12)]) there follows

$$|\Phi_{m+(\nu+1)N}(z, \sigma)| \leqslant c_1(\mathscr{E}) \exp\left(c_2(\mathscr{E}) \sum_{j=m}^{n+\nu N-1} |a_{j+N} - a_j| \right)$$

and (1.5) guarantees the uniform boundedness of the polynomials $\Phi_n(z, \sigma)$ on the subarcs $\{e^{i\varphi} : \varphi \in \mathscr{E}\}$. Thus relation (4.1) is proved.

To finish the proof we have to show that σ is absolutely continuous on $E_l \backslash \{\psi_1, ..., \psi_{N-l}\}$ and that the absolutely continuous part f is positive and continuous there. Since the reflection coefficients of the orthogonal polynomials $P_n(z, \sigma)$ and of the polynomials of the second kind $\Omega_n(z, \sigma)$ only differ by sign and because of the symmetric definition of the polynomials $\mathscr{T}$, $\mathscr{U}$, and R, the orthonormalized polynomials of the second kind

$$\Psi_n(z, \sigma) := \frac{\Omega_n(z, \sigma)}{\sqrt{d_n}}$$

are uniformly bounded on compact subsets of $\{e^{i\varphi} : \varphi \in \operatorname{int} E_l \backslash \{\psi_1, ..., \psi_{N-l}\}\}$, in the same way as the $\Phi_n(z, \sigma)$'s. Now we can apply [9, Lemma 1], which says that σ is absolutely continuous on closed subsets of $\operatorname{int} E_l \backslash \{\psi_1, ..., \psi_{N-l}\}$. The positivity of f on $\operatorname{int} E_l \backslash \{\psi_1, ..., \psi_{N-l}\}$ follows also from the boundedness of the orthonormal polynomials and from [9, Lemma 2]. Finally, using the representation (3.2) of the ϑ_n's it follows from the first statement of Corollary 2 in [15] that $(\vartheta_n f)$ converges weakly to r on compact subsets of $E_l \backslash \{\psi_1, ..., \psi_{N-l}\}$ and thus by (4.1) and the continuity of ϑ and r the assertion is proved. ∎

Proof of Corollary 1. By [7, Theorem 19.1] the relation $\lim_{n\to\infty} a_n = 0$ implies that $\operatorname{supp}(\sigma) = [0, 2\pi]$. The comparison sequence of reflection coefficients $\{a_n^0\}$ is now the constant zero sequence. If we consider this sequence to be periodic with length of period N we get

$$\mathscr{T}(z) = z^N + 1 \qquad \text{and} \qquad R(z) = (z^N - 1)^2, \qquad \mathscr{U}(z) \equiv 1,$$

i.e., $N = l$, which gives

$$r(\varphi) = 2 \sin \frac{N}{2} \varphi.$$

Now all the assertions follow from the proof of Theorem 1 which also holds true for the "limit"-case, i.e., when the arcs form the whole unit circle. ∎

Proof of Theorem 2. Assumption (3.7) guarantees the uniform boundedness of the orthonormal polynomials $\Phi_n(z, \sigma)$ and of the second kind polynomials $\Psi_n(z, \sigma)$ on compact subsets of the arcs Γ_{E_l}; cf. [12, Lemma 3.1]. This shows the uniform convergence in (3.8) and moreover, as in the proof of Theorem 1, the absolute continuity of σ and the positivity of f.

In order to prove relation (3.9), which also implies immediately the continuity of f on int E_l, let us start with the following settings:

$$\Delta_n(z) := \tfrac{1}{2}(\Phi_n^*(z, \sigma)\, \mathscr{G}_n(z, \sigma_0) - z\Phi_n(z, \sigma)\, \mathscr{H}_n(z, \sigma_0)), \qquad n \in \mathbb{N}, \quad (4.3)$$

where $\mathscr{G}_n(z, \sigma_0)$ and $\mathscr{H}_n(z, \sigma_0)$ are the functions of the second kind, given by

$$\begin{aligned} \mathscr{G}_n(z, \sigma_0) &:= \frac{1}{2\pi z^n} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, \Phi_n(e^{i\varphi}, \sigma_0)\, d\sigma_0(\varphi) \\ \mathscr{H}_n(z, \sigma_0) &:= \frac{1}{2\pi z} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, \overline{\Phi_n(e^{i\varphi}, \sigma_0)}\, d\sigma_0(\varphi). \end{aligned} \quad (4.4)$$

Since σ_0 is a "periodic" measure, i.e., it corresponds to periodic reflection coefficients $\{a_n^0\}$, we also have the following representations; compare [12, formula (3.7)]:

$$\begin{aligned} \mathscr{G}_n(z, \sigma_0) &= \frac{1}{z^n V(z)\, A(z)} \left(\frac{\mathscr{T}(z) - \sqrt{R(z)}\, \mathscr{U}(z)}{L} \right)^{\nu} \\ &\quad \times (\sqrt{R(z)}\, \Phi_m(z, \sigma_0) - \mathscr{U}(z)\, \mathscr{Q}_{m+l}(z, \sigma_0)) \\ \mathscr{H}_n(z, \sigma_0) &= \frac{1}{z^{n+1} V(z)\, A(z)} \left(\frac{\mathscr{T}(z) - \sqrt{R(z)}\, \mathscr{U}(z)}{L} \right)^{\nu} \\ &\quad \times (\sqrt{R(z)}\, \Phi_m^*(z, \sigma_0) - \mathscr{U}(z)\, \mathscr{Q}_{m+l}^*(z, \sigma_0)), \end{aligned} \quad (4.5)$$

$n = \nu N + m \in \mathbb{N}_0$. Here, the polynomials $\mathscr{Q}_{m+l}(z, \sigma_0)$ are given by

$$\mathscr{U}(z)\, \mathscr{Q}_{m+l}(z, \sigma_0) := L\Phi_{m+N}(z, \sigma_0) - \mathscr{T}(z)\, \Phi_m(z, \sigma_0), \qquad m \in \mathbb{N}_0. \quad (4.6)$$

The reason for these definitions is that under the assumption (3.7) Theorem 3.3 in [10] together with (2.9) gives

$$|\lim_{n \to \infty} \Delta_n(e^{i\varphi})|^2 = \frac{r(\varphi)}{\mathscr{V}(\varphi)\, \mathscr{A}(\varphi)\, f(\varphi)}, \qquad \varphi \in E_l. \quad (4.7)$$

To prove (3.9), we have to give a more explicit representation of the functions Δ_n: Let the selfreversed polynomial B be given as in the theorem. Then it can be shown that the polynomial $\mathcal{Q}_{n+l}$ from (4.6) is of the form

$$\mathcal{Q}_{n+l}(z, \sigma_0) = -V(z)[A(z)\,\Psi_n(z, \sigma_0) + B(z)\,\Phi_n(z, \sigma_0)],$$

where $\Psi_n(z, \sigma_0) := \Omega_n(z, \sigma_0)/\sqrt{d_n^0}$, and we obtain

$$\begin{aligned}\Delta_n(z) &= \frac{1}{2z^n V(z)\,A(z)}\,[\sqrt{R(z)}\,(\Phi_n(z, \sigma_0)\,\Phi_n^*(z, \sigma) - \Phi_n^*(z, \sigma_0)\,\Phi_n(z, \sigma))\\ &\quad - (\mathcal{Q}_{n+1}(z, \sigma_0)\,\Phi_n^*(z, \sigma) - \mathcal{Q}_{n+1}^*(z, \sigma_0)\,\Phi_n(z, \sigma))]\\ &= \frac{1}{2z^n V(z)\,A(z)}\,[(V(z)\,B(z) + \sqrt{R(z)})(\Phi_n(z, \sigma_0)\,\Phi_n^*(z, \sigma)\\ &\quad - \Phi_n^*(z, \sigma_0)\,\Phi_n(z, \sigma))\\ &\quad + V(z)\,A(z)(\Psi_n(z, \sigma_0)\,\Phi_n^*(z, \sigma) + \Psi_n^*(z, \sigma_0)\,\Phi_n(z, \sigma))].\end{aligned}$$

Now we define the functions

$$\mathcal{S}_{1,n}(z) := \frac{1}{2iz^n}\,(\Phi_n(z, \sigma_0)\,\Phi_n^*(z, \sigma) - \Phi_n^*(z, \sigma_0)\,\Phi_n(z, \sigma))$$

$$\mathcal{S}_{2,n}(z) := \frac{1}{2z^n}\,(\Psi_n(z, \sigma_0)\,\Phi_n^*(z, \sigma) + \Psi_n^*(z, \sigma_0)\,\Phi_n(z, \sigma)),$$

which indeed coincide with the functions given in (3.5) and (3.6). This can be seen in a similar way as we proceeded with the function Θ_n in (3.3) by expanding in a series of orthonormal polynomials (compare also [10, Lemma 2.1]). Now we see that $\mathcal{S}_{1,n}(e^{i\varphi})$ and $\mathcal{S}_{2,n}(e^{i\varphi})$ are real trigonometric polynomials and we can write

$$\Delta_n(z) = \left(\frac{V(z)\,B(z) + \sqrt{R(z)}}{V(z)\,A(z)}\right) i\mathcal{S}_{1,n}(z) + \mathcal{S}_{2,n}(z). \tag{4.8}$$

Using relations (4.8), (3.8), and the definition of VA in the line after (2.9), it is not difficult to derive the following identities from (4.7):

$$\begin{aligned}\frac{r(\varphi)}{f(\varphi)} &= iz^{-l/2}\,V(z)\,A(z)\,\left|\frac{V(z)\,B(z) + \sqrt{R(z)}}{V(z)\,A(z)}\,i\mathcal{S}_1(z) + \mathcal{S}_2(z)\right|^2\\ &= iz^{-l/2}\,V(z)\,A(z)\left[\left(\mathcal{S}_2(z) + \frac{iB(z)\,\mathcal{S}_1(z)}{A(z)}\right)^2 + \frac{R(z)\,\mathcal{S}_1^2(z)}{V^2(z)\,A^2(z)}\right],\end{aligned}$$

$z=e^{i\varphi}$, $\varphi \in \operatorname{int} E_l$. Here, we have made use of the fact that $iB(z)/A(z)$ and $\sqrt{R(z)}/V(z)\,A(z)$ are real on Γ_{E_l}. Hence,

$$\frac{r(\varphi)}{f(\varphi)}=iz^{-l/2}\left[\frac{R(z)-V^2(z)\,B^2(z)}{V(z)\,A(z)}\,\mathscr{S}_1^2(z)+2iV(z)\,B(z)\,\mathscr{S}_1(z)\,\mathscr{S}_2(z)\right.$$
$$\left.+\,V(z)\,A(z)\,\mathscr{S}_2^2(z)\right]$$

and assertion (3.9) follows from the representations of the polynomials R, VA and VB in terms of the polynomials $P_N(z, \sigma_0)$ and $\Omega_N(z, \sigma_0)$. ▮

Proof of Theorem 3. Some of the ideas used in the following proof can be found in [8, pp. 38–40]. Let

$$K_n(z, \xi, \sigma):=\sum_{k=0}^{n} \Phi_k(z, \sigma)\,\overline{\Phi_k(\xi, \sigma)}$$

be the reproducing kernel function, also denoted by Christoffel function, corresponding to the measure σ. By its known reproducing property we can write

$$\begin{aligned}\Phi_n(z, \mu)&=\frac{1}{2\pi}\int_0^{2\pi} \Phi_n(\xi, \mu)\,K_n(z, \xi, \sigma)\,d\sigma(\xi)\\&=\frac{1}{2\pi c}\int_0^{2\pi} \Phi_n(\xi, \mu)\,K_n(z, \xi, \sigma)\,d\mu(\xi)\\&\quad-\sum_{j=1}^{M} \lambda_j \Phi_n(e^{i\alpha_j}, \mu)\,K_n(z, e^{i\alpha_j}, \sigma).\end{aligned}$$

Let κ_n denote the leading coefficient of the orthonormal polynomials (recall (1.3), i.e., $\kappa_n=1/\sqrt{d_n}$). Then orthogonality yields

$$\Phi_n(z, \mu)=\frac{\kappa_n(\sigma)}{c\,\kappa_n(\mu)}\,\Phi_n(z, \sigma)-\sum_{j=1}^{M} \lambda_j \Phi_n(e^{i\alpha_j}, \mu)\,K_n(z, e^{i\alpha_j}, \sigma). \tag{4.9}$$

Comparing the leading coefficients in (4.9) gives the identity

$$\frac{\kappa_n(\mu)}{\kappa_n(\sigma)}=\frac{\kappa_n(\sigma)}{c\,\kappa_n(\mu)}-\sum_{j=1}^{M} \lambda_j \Phi_n(e^{i\alpha_j}, \mu)\,\overline{\Phi_n(e^{i\alpha_j}, \sigma)}. \tag{4.10}$$

Now recall that we have seen in the proof of Theorem 1 that by the location of the points α_j the sequence $\{\Phi_n(e^{i\alpha_j}, \sigma)\}$ is uniformly bounded for all n, $j \in \mathbb{N}$. Further, the α_j's are mass points of μ. Thus, it is well known that

$$\sum_{n=0}^{\infty} |\Phi_n(e^{i\alpha_j}, \mu)|^2 < \infty$$

and consequently

$$\lim_{n \to \infty} \Phi_n(e^{i\alpha_j}, \mu) = 0 \qquad \text{for all} \quad j \in \{1, ..., M\}.$$

Since all the λ_j's are nonnegative and summable, it is not difficult to see that the sum in (4.10) tends to zero as $n \to \infty$. Hence,

$$\lim_{n \to \infty} \left(\frac{\kappa_n(\mu)}{\kappa_n(\sigma)} - \frac{\kappa_n(\sigma)}{c\, \kappa_n(\mu)} \right) = 0,$$

which implies that

$$\lim_{n \to \infty} \frac{\kappa_n(\mu)}{\kappa_n(\sigma)} = \frac{1}{\sqrt{c}}. \tag{4.11}$$

Next, we consider representation (4.9) once again, this time at the point $z = 0$. Using $\Phi_n(0) = \kappa_n a_{n-1}$, we get

$$\kappa_n(\mu)\, a_{n-1}(\mu) = \frac{\kappa_n^2(\sigma)}{c\, \kappa_n(\mu)}\, a_{n-1}(\sigma) - \sum_{j=1}^{M} \lambda_j \Phi_n(e^{i\alpha_j}, \mu)\, K_n(0, e^{i\alpha_j}, \sigma)$$

and from the well known identity, cf. [8, formula (1.7)],

$$K_n(z, \xi, \sigma) = \frac{\Phi_{n+1}^*(z, \sigma)\, \overline{\Phi_{n+1}^*(\xi, \sigma)} - \Phi_{n+1}(z, \sigma)\, \overline{\Phi_{n+1}(\xi, \sigma)}}{1 - z\bar{\xi}},$$

i.e.,

$$K_n(0, e^{i\alpha_j}, \sigma) = \kappa_{n+1}(\sigma)\, \overline{\Phi_{n+1}^*(e^{i\alpha_j}, \sigma)} - \kappa_{n+1}(\sigma)\, a_n(\sigma)\, \overline{\Phi_{n+1}(e^{i\alpha_j}, \sigma)},$$

one obtains

$$\frac{\kappa_n(\mu)}{\kappa_n(\sigma)}\, a_{n-1}(\mu) = \frac{\kappa_n(\sigma)}{c\kappa_n(\mu)}\, a_{n-1}(\sigma) - \frac{\kappa_{n+1}(\sigma)}{\kappa_n(\sigma)} \sum_{j=1}^{M} \lambda_j \Phi_n(e^{i\alpha_j}, \mu)$$
$$\times [\overline{\Phi_{n+1}^*(e^{i\alpha_j}, \sigma)} - a_n(\sigma)\, \overline{\Phi_{n+1}(e^{i\alpha_j}, \sigma)}]. \tag{4.12}$$

Finally, by

$$\frac{\kappa_{n+1}(\sigma)}{\kappa_n(\sigma)} = \frac{1}{\sqrt{1-|a_n(\sigma)|^2}} \leqslant \text{const.} \qquad \text{for all} \quad n \in \mathbb{N}_0$$

and by the same arguments as applied to the identity in (4.10) we see again that the sum in (4.12) tends to zero as $n \to \infty$. Now (4.11) gives

$$\lim_{n \to \infty} (a_n(\mu) - a_n(\sigma)) = 0.$$

This is the assertion. ∎

Proof of Theorem 4. Let $\Upsilon = \overline{\mathbb{C}} \backslash \Gamma$ and let G be a function analytic in Υ. Note that the standard analytic functions defined for the multi-connected region Υ have multi-valued argument in general. The ambiguity of the argument of a function in Υ is characterized as follows (compare [1, p. 237]): Let $\gamma = (\gamma_1, ..., \gamma_l)$ be a vector in $\mathbb{R}^l$. Take the coordinates of γ to be the increments in the argument of a multi-valued function $G(z)$ on marking circuits of the arcs, i.e.,

$$\gamma(G) = \left(\dots, \frac{1}{2\pi} \underset{\Gamma_j}{\triangle} \arg G(z), \dots \right). \tag{4.13}$$

We take the quotient of the function analytic in Υ by the equivalence relation $G_1(z) \approx G_2(z) \Leftrightarrow \gamma(G_1) = \gamma(G_2)$. The classes obtained are denoted by Σ_γ, i.e.,

$$G(z) \in \Sigma_\gamma \qquad \text{if} \quad \gamma = \left(\dots, \frac{1}{2\pi} \underset{\Gamma_j}{\triangle} \arg G(z), \dots \right). \tag{4.14}$$

Next, let Y be the conformal mapping of Υ onto the exterior of the unit disk, i.e.,

$$Y(z) = \exp(g(z, \infty) + i\tilde{g}(z, \infty)),$$

where $g(z, \infty)$ is Green's function for the set $\overline{\mathbb{C}} \backslash \Gamma$ with pole at ∞ and $\tilde{g}(z, \infty)$ is a harmonic conjugate. Further, let us set

$$\Sigma_n := -n\Sigma_{\gamma(Y)}. \tag{4.15}$$

Note that by definition (4.13)

$$\gamma(Y) = (\omega(\Gamma_1, \infty), ..., \omega(\Gamma_l, \infty)), \tag{4.16}$$

see e.g. [18, p. 141], where $\omega(\Gamma_j, \infty)$ is the harmonic measure at $z=\infty$ of the jth arc Γ_j. Furthermore, for $\rho \in L_1(\Gamma)$ let $H_2(\Upsilon, \rho, \Sigma_\gamma)$ be the set of functions G from Σ_γ which are everywhere analytic on Υ and for which $|G(z)^2 \mathscr{R}(z)|$ has a harmonic majorant. Here, $\mathscr{R}(z)$ is the analytic function without zeros or poles in Υ whose modulus on Υ is single-valued and which takes the value $\rho(\xi)$ on Γ (see e.g. [18, p. 155] or [1, p. 237]).

For weight-functions ρ satisfying the condition

$$\oint_\Gamma \log \rho(\xi) \frac{\partial}{\partial n_\xi} g(\xi)\, |d\xi| > -\infty \tag{4.17}$$

Widom has given the following asymptotic representation of the monic polynomials $Q_n(z)$ of degree n orthogonal with respect to $\rho(\xi)\,|d\xi|$ on Γ [18, Theorem 12.3]:

$$Q_n(z)\, C(\Gamma)^{-n}\, \Upsilon^{-n}(z) \sim G_n(z) \qquad \text{for} \quad z \in K \subset \Upsilon, \tag{4.18}$$

K compact and $C(\Gamma)$ the logarithmic capacity of Γ, where $G_n \in H_2(\Upsilon, \rho, \Sigma_n)$ is the unique solution of the following extremal problem:

$$\nu(\rho, \Sigma_n) = \inf_{G \in H_2(\Upsilon, \rho, \Sigma_n)} \int_\Gamma |G(\xi)|^2 \rho(\xi)\, |d\xi|, \tag{4.19}$$

hence,

$$\nu(\rho, \Sigma_n) = \int_\Gamma |G_n(\xi)|^2 \rho(\xi)\, |d\xi|.$$

Now, in the case under consideration we have $\xi = e^{i\varphi}$, $f(\varphi) = \rho(\xi)$ and $|d\xi| = d\varphi$. Furthermore, let us note that $(\partial/\partial n_\xi)\, g(\xi)\, |d\xi|$ can be given explicitly. Indeed, in the proof of Lemma 2 in [14] (the notation in [14] is slightly different from that one here, R in this paper corresponds to R^0 from [14]) we have demonstrated that

$$\frac{\partial}{\partial n_\xi} g(\xi)\, |d\xi| = \left| \frac{S_l(e^{i\varphi})}{\sqrt{R(e^{i\varphi})}} \right| d\varphi,$$

where $S_l(z)$ is the polynomial of degree l uniquely determined by the conditions $S_l = S_l^*$, $iS_l(0) = \sqrt{R(0)}$, and

$$\int_{\varphi_{2j}}^{\varphi_{2j+1}} \frac{S_l(e^{i\varphi})}{\sqrt{R(e^{i\varphi})}}\, d\varphi = 0 \qquad \text{for} \quad j = 1, ..., l-1$$

Hence, the condition (4.17) becomes

$$\int_{E_l} \log f(\varphi) \left| \frac{S_l(e^{i\varphi})}{\sqrt{R(e^{i\varphi})}} \right| d\varphi > -\infty. \tag{4.20}$$

Since in addition $\Gamma \in \mathscr{P}(N)$, i.e., since (3.12) holds, we have by [14, (5.20) and (5.22)]

$$\tau'(\varphi) = \frac{N}{2} (e^{-i((N-l)/2)\varphi} \mathscr{U}(e^{i\varphi}))(e^{-i(l/2)\varphi} S_l(e^{i\varphi})), \tag{4.21}$$

where τ is defined in (2.7). In view of (4.21) we obtain immediately (compare also [11, Section 3]) that $S_l(e^{i\varphi})$ has exactly one zero in each interval $(\varphi_{2j}, \varphi_{2j+1})$, $j = 1, ..., l-1$, and, using the facts that $\tau(\varphi + 2\pi) = \tau(\varphi)$ if N is even and $\tau(\varphi + 2\pi) = -\tau(\varphi)$ if N is odd, one zero in $(\varphi_{2l}, \varphi_1 + 2\pi)$. Since $R(z)$ is a selfreversed polynomial of degree $2l$ which vanishes exactly at the boundary points $e^{i\varphi_j}$, $j = 1, ..., 2l$, of the arcs, we have

$$e^{-il\varphi} R(e^{i\varphi}) = \text{const} \prod_{j=1}^{2l} \sin\left(\frac{\varphi - \varphi_j}{2}\right).$$

Thus, by the supposed generalized Szegő condition, condition (4.20) and therefore (4.17) is satisfied. Furthermore, $Q_n(z) = P_n(z, \sigma)$ and (4.18) becomes

$$P_n(z, \sigma)\, C(\Gamma)^{-n}\, \Upsilon^{-n}(z) \sim G_n(z) \qquad \text{for} \quad z \in K \subset \Upsilon. \tag{4.22}$$

Moreover, the conformal mapping and the logarithmic capacity are explicitly known [14, see the end of Section 2]:

$$\Upsilon(z) = \left(\frac{\mathscr{T}(z) + \sqrt{R(z)}\, \mathscr{U}(z)}{L} \right)^{1/N}$$

$$C(\Gamma) = \sqrt[N]{L/2},$$

where we used the notation from (3.12). In particular, we have

$$C(\Gamma)^N\, \Upsilon^N(0) = 1. \tag{4.23}$$

Since by assumption $\omega(\Gamma_j, \infty) = k_j/N$, $k_j \in \mathbb{N}$, for all $j = 1, ..., l$, we obtain from (4.13)–(4.16)

$$\Sigma_{k+mN} = \Sigma_k \mod 1 \qquad \text{for all} \quad m \in \mathbb{N} \quad \text{and} \quad k = 0, 1, ..., N-1$$

and therefore, by (4.19) and the uniqueness of the extremal function,

$$G_{k+mN} \equiv G_k \qquad \text{for all} \quad m \in \mathbb{N} \qquad \text{and} \qquad k = 0, ..., N-1. \quad (4.24)$$

Now, we only have to evaluate (4.22) at $z = 0$ (recall $P_n(0, \sigma) = a_{n-1}(\sigma)$) and to apply (4.23) in order to obtain our assertion (3.13) with $a_k^0 = G_k(0)$. ∎

ACKNOWLEDGMENT

We would like to thank Paul Nevai for pointing out some incorrectnesses in the manuscript.

REFERENCES

1. A. I. Aptekarev, Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda lattices, *Math. USSR Sb.* **53** (1986), 233–260.
2. M. Bello and G. López, Ratio and relative asymptotics of polynomials orthogonal on an arc of the unit circle, *J. Approx. Theory* **92** (1998), 216–244.
3. D. Barrios Rolania and G. López, Ratio asymptotics for polynomials on arcs of the unit circle, *Constr. Approx.* **15** (1999), 1–31.
4. Ya. L. Geronimus, On the character of the solutions of the moment problem in the case of a limit-periodic associated fraction, *Izv. Akad. Nauk SSSR Ser. Mat.* **5** (1941), 203–210. [in Russian]
5. Ya. L. Geronimus, On polynomials orthogonal on the circle, on the trigonometric moment-problem and on allied Caratheodory and Schur Functions, *C.R. Acad. Sci. URSS (N.S.)* **29** (1943), 291–295.
6. Ya. L. Geronimus, On polynomials orthogonal on a circle, on a trigonometric moment problem, and on the associated functions of Caratheory's and Schur's type, *Mat. Sb. (N.S.)* **15**(57) (1944), 99–130. [in Russian]
7. Ya. L. Geronimus, Polynomials orthogonal on a circle and their applications, *Amer. Math. Soc. Transl.* **3** (1962), 1–78.
8. Ya. L. Geronimus, "Orthogonal Polynomials," Consultants Bureau, New York, 1961.
9. L. Golinskii, P. Nevai, and W. Van Assche, Perturbation of orthogonal polynomials on an arc of the unit circle, *J. Approx. Theory* **83** (1995), 392–422.
10. F. Peherstorfer and R. Steinbauer, Comparative asymptotics for perturbed orthogonal polynomials, *Trans. Math. Soc.* **348**(4) (1996), 1459–1486.
11. F. Peherstorfer and R. Steinbauer, Orthogonal polynomials on arcs of the unit circle. II. Orthogonal polynomials with periodic reflection coefficients, *J. Approx. Theory* **87** (1996), 60–102.
12. F. Peherstorfer and R. Steinbauer, Asymptotics behaviour of orthogonal polynomials on the unit circle with asymptotically periodic reflection coefficients, *J. Approx. Theory* **88** (1997), 316–353.
13. F. Peherstorfer and R. Steinbauer, Mass-points of of orthogonality measures on the unit circle, *East J. Approx.*, to appear.
14. F. Peherstorfer and R. Steinbauer, Strong asymptotics of orthonormal polynomials with the aid of Green's function, *SIAM J. Math. Anal.*, to appear.

15. F. Peherstorfer and R. Steinbauer, Asymptotic behaviour of orthogonal polynomials on the unit circle with asymptotically periodic reflection coefficients, II. Weak asymptotics, *J. Approx. Theory*, submitted.
16. G. Szegő, "Orthogonal polynomials," 4th ed., Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1975.
17. W. Van Assche, Asymptotics of orthogonal polynomials and three-term recurrences, *in* "Orthogonal polynomials: Theory and Practice" (P. Nevai, Ed.), NATO ASI Ser., Vol. 294, pp. 435–462, Kluwer, Dordrecht, 1990.
18. H. Widom, Extremal polynomials associated with a spectrum of curves in the complex plane, *Adv. Math.* **3** (1969), 127–232.